8 $y = \sin 5x + \cos 3x$

$\frac{dy}{dx} = 5\cos 5x - 3\sin 3x$

At $(\pi, -1)$, $\frac{dy}{dx} = 5\cos 5\pi - 3\sin 3\pi$

$= 5 \times (-1) - 3 \times 0 = -5$

Equation of tangent is $y - (-1) = -5(x - \pi)$

or $y = -5x + 5\pi - 1$

9 $y = 2x^2 - \sin x$

$\frac{dy}{dx} = 4x - \cos x$

When $x = \pi$, $y = 2\pi^2$ and

$\frac{dy}{dx} = 4\pi - \cos \pi = 4\pi + 1$

Gradient of normal is $-\frac{1}{4\pi + 1}$

Equation of normal is

$y - 2\pi^2 = -\frac{1}{4\pi + 1}(x - \pi)$

Multiplying through by (4π + 1) and rearranging gives

$x + (4\pi + 1)y - \pi(8\pi^2 + 2\pi + 1) = 0$

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9 Let $y = a^{kx}$

Then $y = e^{\ln a^{kx}} = e^{kx \ln a} = e^{(k \ln a)x}$

$\frac{dy}{dx} = (k \ln a) e^{(k \ln a)x} = k \ln a\, e^{kx \ln a}$

$= k \ln a\, e^{\ln a^{kx}} = a^{kx} k \ln a$

11 c $y = \ln(\cos x)^2$

Let $u = \cos x$; then $y = \ln u^2 = 2\ln u$

$\frac{du}{dx} = -\sin x$ and $\frac{dy}{du} = \frac{2}{u}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{2}{u} \times (-\sin x)$$

$$= -2\frac{\sin x}{\cos x} = -2\tan x$$

d $y = \frac{1}{3+\cos 2x}$

Let $u = 3+\cos 2x$; then $y = \frac{1}{u} = u^{-1}$

$\frac{du}{dx} = -2\sin 2x$ and $\frac{dy}{du} = -u^{-2} = -\frac{1}{u^2}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times (-2\sin 2x)$$

$$= \frac{2\sin 2x}{(3+\cos 2x)^2}$$

e $y = \sin\left(\frac{1}{x}\right)$

Let $u = \frac{1}{x}$; then $y = \sin u$

$\frac{du}{dx} = -\frac{1}{x^2}$ and $\frac{dy}{du} = \cos u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times \left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

12 $y = \frac{4}{(2-4x)^2}$

Let $u = 2-4x$; then $y = \frac{4}{u^2} = 4u^{-2}$

$\frac{du}{dx} = -4$ and $\frac{dy}{du} = -8u^{-3} = -\frac{8}{u^3}$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{8}{u^3} \times (-4) = \frac{32}{(2-4x)^3}$$

When $x = 3$, $y = \frac{4}{(-10)^2} = 0.04$

and $\frac{dy}{dx} = \frac{32}{(-10)^3} = -0.032$

Equation of normal at A is

$$y - 0.04 = \frac{1}{0.032}(x-3)$$

Multiplying through by 100 and rearranging gives

$100y - 4 = 3125x - 9375$

$3125x - 100y - 9371 = 0$

13 $y = 3^{x^3}$

Let $u = x^3$; then $y = 3^u$

$\frac{du}{dx} = 3x^2$ and $\frac{dy}{du} = 3^u \ln 3$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3^u \ln 3 \times 3x^2 = 3x^2 3^{x^3} \ln 3$$

When $x = 1$, $\frac{dy}{dx} = 3 \times 1^2 \times 3^{1^3} \times \ln 3 = 9\ln 3$

6 $y = x^2 \cos(x^2)$

Let $u = x^2$ and $v = \cos(x^2)$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -2x\sin(x^2)$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = x^2\left(-2x\sin(x^2)\right) + \cos(x^2) \times 2x$$
$$= 2x\left(\cos(x^2) - x^2\sin(x^2)\right)$$

When $x = \frac{\sqrt{\pi}}{2}$,

$$\frac{dy}{dx} = \sqrt{\pi}\left(\cos\frac{\pi}{4} - \frac{\pi}{4}\sin\frac{\pi}{4}\right)$$
$$= \sqrt{\pi}\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) = \sqrt{\frac{\pi}{2}}\left(1 - \frac{\pi}{4}\right)$$

Equation of tangent at $P\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ is

$$y - \frac{\pi\sqrt{2}}{8} = \sqrt{\frac{\pi}{2}}\left(1 - \frac{\pi}{4}\right)\left(x - \frac{\sqrt{\pi}}{2}\right)$$

$$8y - \pi\sqrt{2} = 4\sqrt{2\pi}\left(1 - \frac{\pi}{4}\right)\left(x - \frac{\sqrt{\pi}}{2}\right)$$

$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi)\left(x - \frac{\sqrt{\pi}}{2}\right)$$

$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi)x - \frac{\pi\sqrt{2}}{2}(4 - \pi)$$

$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2} + \frac{\pi\sqrt{2}}{2}(4 - \pi) = 0$$

$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2}\left(\frac{\pi - 2}{2}\right) = 0$$

This is in the form $ax + by + c = 0$ with

$a = \sqrt{2\pi}(\pi - 4)$, $b = 8$ and $c = -\pi\sqrt{2}\left(\frac{\pi - 2}{2}\right)$

7 $y = 3x^2(5x-3)^3$

Let $u = 3x^2$ and $v = (5x-3)^3$

$\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = 15(5x-3)^2$

Using $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$\frac{dy}{dx} = 3x^2 \times 15(5x-3)^2 + 6x(5x-3)^3$$

$$= 3x(5x-3)^2\left(15x + 2(5x-3)\right)$$

$$= 3x(5x-3)^2(25x-6)$$

Hence $A = 3, n = 2, B = 25$ and $C = -6$.

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8 **a** $x = \dfrac{e^y}{3+2y}$

When $y = 0$, $x = \dfrac{e^0}{3} = \dfrac{1}{3}$

Coordinates of P are $\left(\dfrac{1}{3}, 0\right)$.

b Let $u = e^y$ and $v = 3+2y$

$\dfrac{du}{dy} = e^y$ and $\dfrac{dv}{dy} = 2$

$\dfrac{dx}{dy} = \dfrac{e^y(3+2y) - 2e^y}{(3+2y)^2} = \dfrac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal to the curve is

$-\dfrac{1}{\frac{dy}{dx}} = -\dfrac{dx}{dy} = -\dfrac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal at $P\left(\dfrac{1}{3}, 0\right)$ is

$-\dfrac{e^0(2\times 0+1)}{3^2} = -\dfrac{1}{9}$

Equation of normal at P is

$y - 0 = -\dfrac{1}{9}\left(x - \dfrac{1}{3}\right)$

$y = -\dfrac{1}{9}x + \dfrac{1}{27}$

This is in the form $y = mx + c$ with

$m = -\dfrac{1}{9}$ and $c = \dfrac{1}{27}$

9 Let $y = \dfrac{x^4}{\cos 3x}$

Let $u = x^4$ and $v = \cos 3x$

$\dfrac{du}{dx} = 4x^3$ and $\dfrac{dv}{dx} = -3\sin 3x$

Using $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{4x^3\cos 3x - x^4(-3\sin 3x)}{\cos^2 3x}$

$= \dfrac{x^3(4\cos 3x + 3x\sin 3x)}{\cos^2 3x}$

10 a $y = \frac{\quad}{(x-2)^2}$

Let $u = e^{2x}$ and $v = (x-2)^2$

$\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = 2(x-2)$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{2(x-2)^2 e^{2x} - 2e^{2x}(x-2)}{(x-2)^4}$$

$$= \frac{2e^{2x}(x-2)((x-2)-1)}{(x-2)^4}$$

$$= \frac{2e^{2x}(x-3)}{(x-2)^3}$$

So $A = 2$, $B = 1$ and $C = 3$.

b When $x = 1$, $y = e^2$

and $\frac{dy}{dx} = \frac{2e^2(-2)}{-1} = 4e^2$

Equation of tangent is

$y - e^2 = 4e^2(x-1)$

$y = 4e^2x - 3e^2$

11 a $f(x) = \frac{2x}{x+5} + \frac{6x}{x^2+7x+10}$

$$f(x) = \frac{2x}{x+5} + \frac{6x}{(x+2)(x+5)}$$

$$= \frac{2x(x+2)}{(x+2)(x+5)} + \frac{6x}{(x+2)(x+5)}$$

$$= \frac{2x^2+4x+6x}{(x+2)(x+5)} = \frac{2x^2+10x}{(x+2)(x+5)}$$

$$= \frac{2x(x+5)}{(x+2)(x+5)} = \frac{2x}{x+2}$$

In the last line, dividing through by $(x + 5)$ is allowed because $x > 0$ so $x + 5 \neq 0$.

b Let $u = 2x$ and $v = x + 2$

$\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 1$

$$f'(x) = \frac{2(x+2) - 2x}{(x+2)^2} = \frac{4}{(x+2)^2}$$

Hence $f'(3) = \frac{4}{5^2} = \frac{4}{25}$

12 a $\mathrm{f}(x)=\dfrac{2\cos 2x}{\mathrm{e}^{2-x}}$

Let $u=2\cos 2x$ and $v=\mathrm{e}^{2-x}$

$\dfrac{\mathrm{d}u}{\mathrm{d}x}=-4\sin 2x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x}=-\mathrm{e}^{2-x}$

$$\mathrm{f}'(x)=\frac{-4\mathrm{e}^{2-x}\sin 2x-2\cos 2x\,(-\mathrm{e}^{2-x})}{(\mathrm{e}^{2-x})^2}$$

$$=\frac{2\mathrm{e}^{2-x}(\cos 2x-2\sin 2x)}{(\mathrm{e}^{2-x})^2}$$

At stationary points, $\mathrm{f}'(x)=0$

$\cos 2x-2\sin 2x=0$

$2\sin 2x=\cos 2x$

$\therefore\ \tan 2x=\dfrac{1}{2}$

b The range of f(x) is between the y-coordinate of B and the y-coordinate of the right endpoint of the interval.

$\tan 2x=\dfrac{1}{2}\Rightarrow 2x=0.4636$ or 3.6052

$x=0.2318$ or 1.8026

So the x-coordinate of B is 1.8026.

Range of f(x) is

$\mathrm{f}(1.8026)\leqslant y<\mathrm{f}(\pi)$

$-1.47\leqslant y<6.26$ (3 s.f.)

5 a $y = \frac{1}{\cos x \sin x} = \sec x \operatorname{cosec} x$

Let $u = \sec x$ and $v = \operatorname{cosec} x$

$\frac{du}{dx} = \sec x \tan x$ and $\frac{dv}{dx} = -\operatorname{cosec} x \cot x$

Using the product rule,

$$\frac{dy}{dx} = \sec x(-\operatorname{cosec} x \cot x)$$
$$+ \operatorname{cosec} x(\sec x \tan x)$$
$$= -\frac{\cos x}{\cos x \sin x \sin x} + \frac{\sin x}{\sin x \cos x \cos x}$$
$$= -\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}$$

Alternative solution:

$y = \frac{1}{\cos x \sin x} = \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x$

(because $\sin 2x = 2\sin x \cos x$)

$\frac{dy}{dx} = -4\operatorname{cosec} 2x \cot 2x$

b At stationary points $\frac{dy}{dx} = 0$

$\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} = 0$

$\frac{1}{\cos^2 x} = \frac{1}{\sin^2 x}$

$\tan^2 x = 1$

$\tan x = \pm 1$

In the interval $0 < x \leqslant \pi$

there are two solutions, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

So the number of stationary points is 2.

Alternative solution:

$-4\operatorname{cosec} 2x \cot 2x = 0$

$\operatorname{cosec} 2x \neq 0$

but $\cot 2x = 0$ has two solutions

in the interval $0 < x \leqslant \pi$.

So there are 2 stationary points.

11 $x = (\arccos y)^2$

$\sqrt{x} = \arccos y$

$y = \cos(\sqrt{x})$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$; then $y = \cos u$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ and $\frac{dy}{du} = -\sin u$

Using the chain rule,

$\frac{dy}{dx} = -\sin u \times \frac{1}{2\sqrt{x}} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}$

$\sin^2\sqrt{x} + \cos^2\sqrt{x} = 1$

$\sin\sqrt{x} = \sqrt{1 - \cos^2\sqrt{x}}$

$\therefore \frac{dy}{dx} = -\frac{\sqrt{1-\cos^2\sqrt{x}}}{2\sqrt{x}}$

12 a $x = \operatorname{cosec} 5y$

$\frac{dx}{dy} = -5\operatorname{cosec} 5y \cot 5y$

$\frac{dy}{dx} = -\frac{1}{5\operatorname{cosec} 5y \cot 5y}$

b $1 + \cot^2 5y = \operatorname{cosec}^2 5y$

$\cot 5y = \sqrt{\operatorname{cosec}^2 5y - 1} = \sqrt{x^2 - 1}$

$\therefore \frac{dy}{dx} = -\frac{1}{5x\sqrt{x^2-1}}$

7 a $x = 2\sin^2 t,\ y = 2\cot t$

$$\frac{dx}{dt} = 4\sin t\cos t,\ \frac{dy}{dt} = -2\operatorname{cosec}^2 t$$

$$\therefore \frac{dy}{dx} = -\frac{2\operatorname{cosec}^2 t}{4\sin t\cos t} = -\frac{1}{2}\sec t\operatorname{cosec}^3 t$$

b When $t = \frac{\pi}{6},\ x = \frac{1}{2}, y = 2\sqrt{3}$

and $\frac{dy}{dx} = -\frac{1}{2}\times\frac{2}{\sqrt{3}}\times 2^3 = -\frac{8}{\sqrt{3}} = -\frac{8\sqrt{3}}{3}$

Equation of tangent is

$$y - 2\sqrt{3} = -\frac{8\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$\sqrt{3}y - 6 = -8x + 4$$

$$8x + \sqrt{3}y - 10 = 0$$

8 a $x = 4\sin t,\ y = 2\operatorname{cosec} 2t$

$x = 2\sqrt{3} \Rightarrow 4\sin t = 2\sqrt{3}$

$\sin t = \dfrac{\sqrt{3}}{2} \quad \therefore\ t = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$t = \dfrac{\pi}{3} \Rightarrow y = 2\operatorname{cosec}\dfrac{2\pi}{3} = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3},$

which is the y-coordinate of point A.

So $t = \dfrac{\pi}{3}$ at point A.

b $\dfrac{dx}{dt} = 4\cos t,\ \dfrac{dy}{dt} = -4\operatorname{cosec} 2t \cot 2t$

$\therefore\ \dfrac{dy}{dx} = -\dfrac{-4\operatorname{cosec} 2t \cot 2t}{4\cos t}$

$= -\dfrac{\cot 2t \operatorname{cosec} 2t}{\cos t}$

When $t = \dfrac{\pi}{3},\ \dfrac{dy}{dx} = -\dfrac{\cot\dfrac{2\pi}{3}\operatorname{cosec}\dfrac{2\pi}{3}}{\cos\dfrac{\pi}{3}}$

$= -\dfrac{\left(-\dfrac{1}{\sqrt{3}}\right) \times \dfrac{2}{\sqrt{3}}}{\dfrac{1}{2}} = \dfrac{4}{3}$

$\therefore$ gradient of normal is $-\dfrac{3}{4}$

Equation of normal, l, is

$y - \dfrac{4\sqrt{3}}{3} = -\dfrac{3}{4}(x - 2\sqrt{3})$

$12y - 16\sqrt{3} = -9(x - 2\sqrt{3})$

$9x + 12y - 34\sqrt{3} = 0$

9 a $x = t^2 + t,\ y = t^2 - 10t + 5$

$\frac{dx}{dt} = 2t + 1,\ \frac{dy}{dt} = 2t - 10$

$\therefore \frac{dy}{dx} = \frac{2t - 10}{2t + 1}$

When gradient is 2, $\frac{2t - 10}{2t + 1} = 2$

$2t - 10 = 4t + 2 \Rightarrow t = -6$

At P, $x = (-6)^2 - 6 = 30$

and $y = (-6)^2 - 10(-6) + 5 = 101$

Coordinates of P are (30, 101).

9 b Equation of tangent at P is

$y - 101 = 2(x - 30)$

$y = 2x + 41$

c Substituting for y and x in the tangent equation:

$t^2 - 10t + 5 = 2(t^2 + t) + 41$

$t^2 + 12t + 36 = 0$

Discriminant $= 12^2 - 4 \times 36 = 0$

Therefore the curve and the line only intersect once, so the tangent at P does not intersect the curve again.

10 a $x = 2\sin t,\ y = \sqrt{2}\cos 2t$

$$\frac{dx}{dt} = 2\cos t,\ \frac{dy}{dt} = -2\sqrt{2}\sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{-2\sqrt{2}\sin 2t}{2\cos t} = \frac{-\sqrt{2}\times 2\sin t\cos t}{\cos t}$$

$$= -2\sqrt{2}\sin t$$

b When $t = \frac{\pi}{3}$:

$$x = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3},\ y = \sqrt{2}\left(-\frac{1}{2}\right) = -\frac{\sqrt{2}}{2}$$

and $\frac{dy}{dx} = -2\sqrt{2}\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{6}$

Equation of normal at A is

$$y - \left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{\sqrt{6}}(x - \sqrt{3})$$

$$\sqrt{6}y + \sqrt{3} = x - \sqrt{3}$$

$$x - \sqrt{6}y - 2\sqrt{3} = 0$$

c Substituting for y and x in the normal equation:

$$2\sin t - \sqrt{6}\times\sqrt{2}\cos 2t - 2\sqrt{3} = 0$$

$$\sin t - \sqrt{3}\cos 2t - \sqrt{3} = 0$$

$$\sin t - \sqrt{3}(1 - 2\sin^2 t) - \sqrt{3} = 0$$

$$2\sqrt{3}\sin^2 t + \sin t - 2\sqrt{3} = 0$$

$$(2\sin t - \sqrt{3})(\sqrt{3}\sin t + 2) = 0$$

$$\sin t = \frac{\sqrt{3}}{2} \quad \text{or} \quad \sin t = -\frac{2}{\sqrt{3}}$$

(2nd option not possible since $|\sin t| \le 1$)

$$\sin t = \frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

When $t = \frac{2\pi}{3}$:

$$x = 2\sin\frac{2\pi}{3} = \sqrt{3},\ y = \sqrt{2}\cos\frac{4\pi}{3} = -\frac{\sqrt{2}}{2},$$

which is the same as point A, so l does not intersect C other than at point A.

11 a $x = \cos t,\ y = \frac{1}{2}\sin 2t$

$\frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos 2t$

$\therefore \frac{dy}{dx} = -\frac{\cos 2t}{\sin t}$

b When $t = \frac{\pi}{6}$: $x = \frac{\sqrt{3}}{2},\ y = \frac{\sqrt{3}}{4}$

and $\frac{dy}{dx} = -\frac{\frac{1}{2}}{\frac{1}{2}} = -1$

Equation of tangent at A is

$y - \frac{\sqrt{3}}{4} = -\left(x - \frac{\sqrt{3}}{2}\right)$

i.e. $y = -x + \frac{3\sqrt{3}}{4}$

11 c l_1 and l_2 both have gradient -1

$\therefore$ values of t at points where the tangents cut the curve will be solutions to

$-\frac{\cos 2t}{\sin t} = -1$

$1 - 2\sin^2 t = \sin t$

$2\sin^2 t + \sin t - 1 = 0$

$(2\sin t - 1)(\sin t + 1) = 0$

$\sin t = \frac{1}{2}$ or -1

$t = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$

So lines l_1 and l_2 touch the curve when

$t = \frac{5\pi}{6}$ and $t = \frac{3\pi}{2}$.

$t = \frac{5\pi}{6} \Rightarrow x = -\frac{\sqrt{3}}{2},\ y = -\frac{\sqrt{3}}{4}$

Equation of l_1 is

$y - \left(-\frac{\sqrt{3}}{4}\right) = -1\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right)$

i.e. $y = -x - \frac{3\sqrt{3}}{4}$

$t = \frac{3\pi}{2} \Rightarrow x = 0,\ y = 0$

Equation of l_2 is

$y - 0 = -(x - 0)$

i.e. $y = -x$

7 $2x^2+3y^2-x+6xy+5=0$

$$4x+6y\frac{\mathrm{d}y}{\mathrm{d}x}-1+6\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)=0$$

$$(6y+6x)\frac{\mathrm{d}y}{\mathrm{d}x}=1-6y-4x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-6y-4x}{6(x+y)}$$

When $x=1$ and $y=-2$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-6(-2)-4}{6(1-2)}=-\frac{3}{2}$$

Equation of tangent at $(1, -2)$ is

$$y-(-2)=-\frac{3}{2}(x-1)$$

$$2y+4=-3x+3$$

$$3x+2y+1=0$$

10 a $\sin x + \cos y = 0.5$

$\cos x - \sin y \dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{\cos x}{\sin y}$

b At stationary points $\dfrac{dy}{dx} = 0$

$\dfrac{\cos x}{\sin y} = 0$ when $\cos x = 0$

$\therefore x = \pm\dfrac{\pi}{2}$ (in the interval $-\pi < x < \pi$)

When $x = \dfrac{\pi}{2}$, $1 + \cos y = 0.5$

$\cos y = -0.5 \Rightarrow y = \pm\dfrac{2\pi}{3}$

When $x = -\dfrac{\pi}{2}$, $-1 + \cos y = 0.5$

$\cos y = 1.5 \Rightarrow$ no solutions

Therefore the stationary points are
$\left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$ and $\left(\dfrac{\pi}{2}, -\dfrac{2\pi}{3}\right)$.

11 a $ye^{-3x} - 3x = y^2$

$y(-3e^{-3x}) + e^{-3x}\dfrac{dy}{dx} - 3 = 2y\dfrac{dy}{dx}$

$(e^{-3x} - 2y)\dfrac{dy}{dx} = 3(ye^{-3x} + 1)$

$\dfrac{dy}{dx} = \dfrac{3(ye^{-3x} + 1)}{e^{-3x} - 2y}$

b Substitute $x = 0$ and $y = 0$ to give

$\dfrac{dy}{dx} = \dfrac{3(0 \times e^0 + 1)}{e^0 - 2 \times 0} = 3$

Equation of tangent at (0, 0) is

$y - 0 = 3(x - 0)$

$y = 3x$

8 $3^x = y - 2xy$

$$3^x \ln 3 = \frac{dy}{dx} - 2\left(x\frac{dy}{dx} + y\right)$$

$$3^x \ln 3 + 2y = (1-2x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3^x \ln 3 + 2y}{1-2x}$$

Substitute $x = 2$ and $y = -3$ to give

$$\frac{dy}{dx} = \frac{3^2 \ln 3 - 6}{1-4} = 2 - 3\ln 3$$

9 $\ln(y^2) = \frac{1}{2}x\ln(x-1)$

$$2\ln y = \frac{1}{2}x\ln(x-1)$$

$$\frac{2}{y}\frac{dy}{dx} = \frac{1}{2}\left(\left(x \times \frac{1}{x-1}\right) + \ln(x-1)\right)$$

$$\frac{dy}{dx} = \frac{y}{4}\left(\frac{x}{x-1} + \ln(x-1)\right)$$

When $x = 4$,
the equation of the curve gives

$$\ln(y^2) = 2\ln 3 = \ln 9 \Rightarrow y^2 = 9$$

$\therefore\ y = 3$ (because $y > 0$)

Hence $\frac{dy}{dx} = \frac{3}{4}\left(\frac{4}{3} + \ln 3\right) = 1 + \frac{3}{4}\ln 3$

---

11 $y = \frac{1}{3}x^2 \ln x - 2x + 5$

$$\frac{dy}{dx} = \frac{1}{3}x^2\left(\frac{1}{x}\right) + \frac{2}{3}x\ln x - 2 = \frac{x}{3} + \frac{2}{3}x\ln x - 2$$

$$\frac{d^2y}{dx^2} = \frac{1}{3} + \frac{2}{3}(1 + \ln x) = 1 + \frac{2}{3}\ln x$$

C is convex when $\frac{d^2y}{dx^2} > 0$

$$1 + \frac{2}{3}\ln x > 0$$

$$\ln x > -\frac{3}{2}$$

$$x > e^{-\frac{3}{2}}$$