Question 5 (+)**

The area, A km^2, of an oil spillage on the surface of the sea, at time t hours after it was formed, satisfies the differential equation

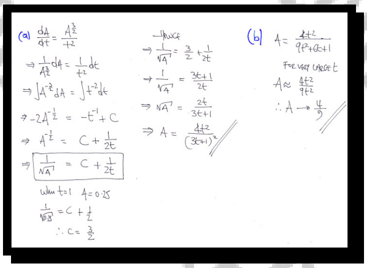
$$\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}, \quad t > 0.$$

When $t = 1$, $A = 0.25$.

a) Find a solution of the differential equation, in the form $A = f(t)$.

b) Determine the largest area that the oil spillage will ever attain.

$$\boxed{A = \frac{4t^2}{(3t+1)^2}}, \quad \boxed{A_{\text{max}} \to \frac{4}{9}}$$



(a) $\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}$

$\Rightarrow \frac{1}{A^{\frac{3}{2}}} dA = \frac{1}{t^2} dt$

$\Rightarrow \int A^{-\frac{3}{2}} dA = \int t^{-2} dt$

$\Rightarrow -2A^{-\frac{1}{2}} = -t^{-1} + C$

$\Rightarrow A^{-\frac{1}{2}} = C + \frac{1}{2t}$

$\Rightarrow \boxed{\frac{1}{\sqrt{A}} = C + \frac{1}{2t}}$

when $t=1$ $A=0.25$

$\frac{1}{\sqrt{0.25}} = C + \frac{1}{2}$

$\therefore C = \frac{3}{2}$

Hence

$\Rightarrow \frac{1}{\sqrt{A}} = \frac{3}{2} + \frac{1}{2t}$

$\Rightarrow \frac{1}{\sqrt{A}} = \frac{3t+1}{2t}$

$\Rightarrow \sqrt{A} = \frac{2t}{3t+1}$

$\Rightarrow A = \frac{4t^2}{(3t+1)^2}$

(b) $A = \frac{4t^2}{9t^2+6t+1}$

For very large t

$A \approx \frac{4t^2}{9t^2}$

$\therefore A \to \frac{4}{9}$

Question 1 (**)

The number of bacterial cells N on a laboratory dish is increasing, so that the <u>hourly</u> rate of increase is 5 times the number of the bacteria present at that time.

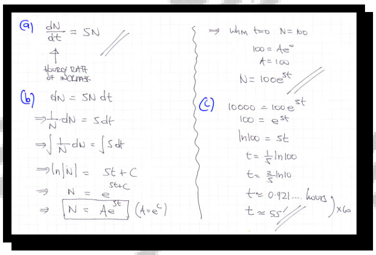
Initially 100 bacteria were placed on the dish.

a) Form a suitable differential equation to model this problem.

b) Find the solution of this differential equation.

c) Find to the nearest minute, the time taken for the bacteria to reach 10000.

$\boxed{\frac{dN}{dt} = 5N}$, $\boxed{N = 100e^{5t}}$, $\boxed{55 \text{ minutes}}$



(a) $\frac{dN}{dt} = 5N$

HOURLY RATE OF INCREASE

(b) $dN = 5N\,dt$

$\Rightarrow \frac{1}{N}\,dN = 5\,dt$

$\Rightarrow \int \frac{1}{N}\,dN = \int 5\,dt$

$\Rightarrow \ln|N| = 5t + C$

$\Rightarrow N = e^{5t+C}$

$\Rightarrow \boxed{N = Ae^{5t}}$ $(A = e^{C})$

$\Rightarrow$ when $t=0$ $N=100$

$100 = Ae^{0}$

$A = 100$

$N = 100e^{5t}$

(c) $10000 = 100e^{5t}$

$100 = e^{5t}$

$\ln 100 = 5t$

$t = \frac{1}{5}\ln 100$

$t = \frac{2}{5}\ln 10$

$t \approx 0.921\ldots$ hours

$t \approx 55'$ $\times 60$

Question 2 (**)

The gradient at any point (x, y) on a curve $y = f(x)$ is proportional to the square root of the y coordinate of that point.

a) Form a suitable differential equation to model this problem.

b) Find a general solution of this differential equation, in terms of suitable constants.

The curve passes though the points $P(4,4)$ and $Q(6,16)$.

c) Find a solution to the differential equation in the form $y = f(x)$.

$\boxed{\frac{dy}{dx} = k\sqrt{y}}$, $\boxed{\sqrt{y} = Ax + B}$, $\boxed{y = (x-2)^2}$

(a) $\frac{dy}{dx} = k\sqrt{y}$

GRADIENT

PROPORTIONAL

SQUARE ROOT OF y COORDINATE

(b) $\frac{1}{\sqrt{y}}\,dy = k\,dx$

$\Rightarrow \int y^{-\frac{1}{2}}\,dy = \int k\,dx$

$\Rightarrow 2y^{\frac{1}{2}} = kx + C$

$\Rightarrow y^{\frac{1}{2}} = Ax + B$

$(4,4) \Rightarrow \sqrt{4} = 4A + B$

$(6,16) \Rightarrow \sqrt{16} = 6A + B$

$4A + B = 2$

$6A + B = 4$

$A = 1$

$B = -2$

$\therefore \sqrt{y} = x - 2$

$y = (x-2)^2$

Question 4 (+)**

Water is leaking out of a tank from a tap which is located 5 cm from the bottom of the tank.

The height of the water, h cm, is decreasing at a rate proportional to square root of the difference of the height of the water and the height of the tap.

a) Model this problem with a differential equation involving h, the time t in minutes and a suitable proportionality constant.

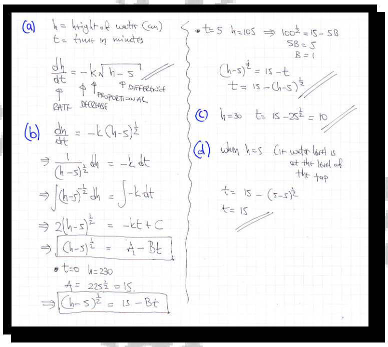
The initial height of the water in the tank is 230 cm and 5 minutes later it has dropped to 105 cm.

b) Find a solution of the differential equation of part **(a)**, in the form $t = f(h)$.

c) Calculate the time taken for the height of the water to fall to 30 cm.

d) State how many minutes it takes for the tank to stop leaking.

$\boxed{\dfrac{dh}{dt} = -k\sqrt{h-5}}$, $\boxed{t = 15 - \sqrt{h-5}}$, $\boxed{10 \text{ minutes}}$, $\boxed{15 \text{ minutes}}$



(a) h = height of water (cm)
t = time in minutes

$\dfrac{dh}{dt} = -k\sqrt{h-5}$

RATE DECREASE PROPORTIONAL DIFFERENCE

(b) $\dfrac{dh}{dt} = -k(h-5)^{\frac{1}{2}}$

$\Rightarrow \dfrac{1}{(h-5)^{\frac{1}{2}}}\,dh = -k\,dt$

$\Rightarrow \int (h-5)^{-\frac{1}{2}}\,dh = \int -k\,dt$

$\Rightarrow 2(h-5)^{\frac{1}{2}} = -kt + C$

$\Rightarrow \boxed{(h-5)^{\frac{1}{2}} = A - Bt}$

• $t=0$ $h=230$

$A = 225^{\frac{1}{2}} = 15$

$\Rightarrow \boxed{(h-5)^{\frac{1}{2}} = 15 - Bt}$

• $t=5$ $h=105$ $\Rightarrow 100^{\frac{1}{2}} = 15 - 5B$

$5B = 5$

$B = 1$

$(h-5)^{\frac{1}{2}} = 15 - t$

$t = 15 - (h-5)^{\frac{1}{2}}$

(c) $h=30$ $t = 15 - 25^{\frac{1}{2}} = 10$

(d) when $h=5$ (the water level is at the level of the tap)

$t = 15 - (5-5)^{\frac{1}{2}}$

$t = 15$

Question 8 (***)

A species of tree is growing in height and the typical maximum height it can reach in its lifetime is 12 m.

The rate of growth of its height, H m, is proportional to the difference between its height and the maximum height it can reach.

When a tree of this species was planted, it was 1 m in height and at that instant the tree was growing at the rate of 0.1 m per month.

a) Show clearly that

$$110\frac{dH}{dt} = 12 - H,$$

where t is the time, measured in months, since the tree was planted.

b) Determine a simplified solution for the above differential equation, giving the answer in the form $H = f(t)$.

c) Find, correct to 2 decimal places, the height of the tree after 5 **years**.

d) Calculate, correct to the nearest year, the number of years it will take for the tree to reach a height of 11 m.

$\boxed{}$, $\boxed{H = 12 - 11e^{-\frac{1}{110}t}}$, $\boxed{5.62 \text{ m}}$, $\boxed{t = 110\ln 11 \text{ months} \approx 22 \text{ years}}$

a) FORMING THE DIFFERENTIAL EQUATION FROM THE INFORMATION GIVEN

$$\underbrace{\frac{dH}{dt}}_{\text{RATE}} = \underbrace{+}_{\text{INCREASE}}\underbrace{k}_{\text{PROPORTIONAL}}(\overbrace{12}^{\text{MAX HEIGHT}} - \underbrace{H}_{\text{HEIGHT}})$$

H = height of tree (m)
t = time (months)
$H_{MAX} = 12\text{ m}$
$t=0$
$H=1$
$\left.\frac{dH}{dt}\right|_{\substack{t=0\\H=1}} = 0.1$

- APPLY CONDITION $\left.\frac{dH}{dt}\right|_{H=1} = 0.1$

$0.1 = k(12-1)$
$k = \frac{1}{110}$

- $\frac{dH}{dt} = \frac{1}{110}(12-H)$

$110\frac{dH}{dt} = 12 - H$ // AS REQUIRED

b) SOLVING THE O.D.E. BY SEPARATING VARIABLES

$\Rightarrow 110\,dH = (12-H)\,dt$
$\Rightarrow \frac{110}{12-H}\,dH = 1\,dt$
$\Rightarrow \int \frac{110}{12-H}\,dH = \int 1\,dt$
$\Rightarrow -110\ln|12-H| = t + C$
$\Rightarrow \ln|12-H| = -\frac{1}{110}t + C$
$\Rightarrow 12-H = e^{-\frac{1}{110}t + C}$
$\Rightarrow 12-H = e^{-\frac{1}{110}t} \times e^{C}$
$\Rightarrow 12-H = Ae^{-\frac{1}{110}t} \quad (A = e^{C})$
$\Rightarrow H = 12 + Ae^{-\frac{1}{110}t}$

APPLY THE CONDITION $t=0$ $H=1$

$\Rightarrow 1 = 12 + A$
$\Rightarrow A = -11$

$\therefore H = 12 - 11e^{-\frac{1}{110}t}$ //

c) WHEN $t=60$ (5 YEARS = 60 MONTHS)

$\Rightarrow H = 12 - 11e^{-\frac{1}{110}\times 60}$
$\Rightarrow H = 12 - 11e^{-\frac{6}{11}}$
$\Rightarrow H \simeq 5.62\text{ m}$ //

d) WHEN $H=11$

$\Rightarrow 11 = 12 - 11e^{-\frac{1}{110}t}$
$\Rightarrow 11e^{-\frac{1}{110}t} = 1$
$\Rightarrow e^{-\frac{1}{110}t} = \frac{1}{11}$
$\Rightarrow e^{\frac{1}{110}t} = 11$
$\Rightarrow \frac{1}{110}t = \ln 11$
$\Rightarrow t = 110\ln 11 \approx 263.76\ldots \text{ months} \xrightarrow{\div 12} \approx 22 \text{ YEARS}$

Question 27 (****+)

In a cold winter morning when the temperature of the air is $10°\text{C}$, Ben the builder pours a cup of coffee out of his flask.

Let x be the temperature of the coffee, in $°\text{C}$, t minutes after it was poured.

The rate at which the temperature of the coffee is decreasing is proportional to the square of the difference between the temperature of the coffee and the air temperature.

The initial temperature of the coffee is $80°\text{C}$ and ten minutes later the temperature of the coffee has dropped to $40°\text{C}$.

By forming and solving a suitable differential equation show that

$$x = \frac{20t + 1200}{2t + 15},$$

and hence find after how many minutes the coffee will have a temperature of $20°\text{C}$.

$\boxed{}$, $\boxed{t = 45}$

FORMING A DIFFERENTIAL EQUATION

x = COFFEE TEMPERATURE (°C)
t = TIME (MINUTES)
$t = 0$, $x = 80$
$t = 10$, $x = 40$

$$\frac{dx}{dt} = -k(x-10)^2$$

RATE
DECREASING
PROPORTIONAL
"DIFFERENCE ... SQUARED"

SOLVING BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{dx}{dt} = -k(x-10)^2$$
$$\Rightarrow dx = -k(x-10)^2\,dt$$
$$\Rightarrow \frac{1}{(x-10)^2}\,dx = -k\,dt$$
$$\Rightarrow \int (x-10)^{-2}\,dx = \int -k\,dt$$
$$\Rightarrow -(x-10)^{-1} = -kt + C$$
$$\Rightarrow \frac{1}{x-10} = At + B$$

APPLY THE CONDITIONS GIVEN

$$t=0,\ x=80 \Rightarrow \frac{1}{70} = B$$
$$\Rightarrow \frac{1}{x-10} = At + \frac{1}{70}$$
$$t=10,\ x=40 \Rightarrow \frac{1}{30} = 10A + \frac{1}{70}$$
$$\Rightarrow A = \frac{1}{525}$$
$$\Rightarrow \frac{1}{x-10} = \frac{1}{525}t + \frac{1}{70} \quad \times 1050$$
$$\Rightarrow \frac{1050}{x-10} = 2t + 15$$
$$\Rightarrow \frac{1050}{2t+15} = x - 10$$
$$\Rightarrow x = \frac{1050}{2t+15} + 10$$
$$\Rightarrow x = \frac{1050 + 10(2t+15)}{2t+15}$$
$$\Rightarrow x = \frac{20t + 1200}{2t+15}$$

AS REQUIRED

FINALLY WHEN $x = 20$

$$\frac{1050}{x-10} = 2t + 15 \quad \text{(FROM EARLIER)}$$
$$\frac{1050}{20-10} = 2t + 15$$
$$105 = 2t + 15$$
$$90 = 2t$$
$$t = 45$$

Question 9 **(****+)**

A water tank has the shape of a hollow inverted hemisphere with a radius of 1 m.

It can be shown by calculus that when the depth of the water in the tank is h m, its volume, V m^3, is given by the formula

$$V = \frac{1}{3}\pi h^2 (3-h).$$

Water is leaking from a hole at the bottom of the tank, in m^3 per hour, at a rate proportional to the volume of the water left in the tank at that time.

a) Show clearly that

$$\frac{dh}{dt} = -\frac{kh(3-h)}{3(2-h)},$$

where k is a positive constant.

The water tank is initially full.

b) Solve the differential equation to show further that

$$3h^2 - h^3 = 2e^{-kt}.$$

$\boxed{}$, $\boxed{\text{proof}}$

a) FORMING AN O.D.E.

$$\Rightarrow \frac{dV}{dt} = -kV$$

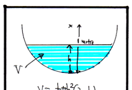
$$\Rightarrow \frac{dV}{dh} \times \frac{dh}{dt} = -kV$$

$$\Rightarrow \cancel{\pi}(2h-h^2)\frac{dh}{dt} = -k \times \frac{1}{3}\cancel{\pi}h^2(3-h)$$

$$\Rightarrow h(2-h)\frac{dh}{dt} = -\frac{1}{3}kh^2(3-h)$$

$$\Rightarrow (2-h)\frac{dh}{dt} = -\frac{1}{3}kh(3-h)$$

$$\Rightarrow \frac{dh}{dt} = -\frac{kh(3-h)}{3(2-h)}$$ // AS REQUIRED



$$V = \frac{1}{3}\pi h^2(3-h)$$

$$V = \pi\left(h^2 - \frac{1}{3}h^3\right)$$

$$\frac{dV}{dh} = \pi\left(2h - h^2\right)$$

b) SEPARATING VARIABLES

$$\Rightarrow \frac{3(2-h)}{h(3-h)}\,dh = -k\,dt$$

$$\Rightarrow \int \frac{3(2-h)}{h(3-h)}\,dh = \int -k\,dt.$$

OBTAIN THE PARTIAL FRACTIONS

$$\frac{3(2-h)}{h(3-h)} \equiv \frac{A}{h} + \frac{B}{3-h}$$

$$3(2-h) \equiv A(3-h) + Bh$$

- IF $h=3$: $-3 = 3B$, $B = -1$
- IF $h=0$: $6 = 3A$, $A = 2$

$$\frac{3(2-h)}{h(3-h)} = \frac{2}{h} - \frac{1}{3-h}$$

CARRYING OUT THE INTEGRATIONS

$$\Rightarrow \int \frac{2}{h} - \frac{1}{3-h}\,dh = -kt + C$$

$$\Rightarrow 2\ln|h| + \ln|3-h| = -kt + C$$

$$\Rightarrow \ln h^2 + \ln|3-h| = -kt + C$$

$$\Rightarrow \ln|h^2(3-h)| = -kt + C$$

$$\Rightarrow h^2(3-h) = e^{-kt+C}$$

$$\Rightarrow 3h^2 - h^3 = e^{-kt} \times e^{C}$$

$$\Rightarrow 3h^2 - h^3 = Ae^{-kt}$$

APPLY THE INITIAL CONDITION $t=0$, $h=1$ (INITIALLY FULL)

$$\Rightarrow 3 - 1 = Ae^0$$

$$\Rightarrow A = 2$$

$$\therefore 3h^2 - h^3 = 2e^{-kt}$$ // AS REQUIRED

Question 13 (****+)

Water is pouring into a long vertical cylinder at a constant rate of $2400 \text{ cm}^3\text{s}^{-1}$ and leaking out of a hole at the base of the cylinder at a rate proportional to the square root of the height of the water already in the cylinder.

The cylinder has constant cross sectional area of 4800 cm^2.

a) Show that, if H is the height of the water in the cylinder, in cm, at time t seconds, then

$$\frac{dH}{dt} = \frac{1}{2} - B\sqrt{H},$$

where B is positive constant.

The cylinder was initially empty and when the height of the water in the cylinder reached 16 cm water was **leaking out of the hole**, at the rate of $120 \text{ cm}^3\text{s}^{-1}$.

b) Show clearly that

$$\frac{dH}{dt} = \frac{80 - \sqrt{H}}{160}.$$

c) Use the substitution $u = 80 - \sqrt{H}$, to find

$$\int \frac{1}{80 - \sqrt{H}} \, dH.$$

[continues overleaf]

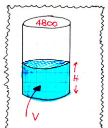
[continued from overleaf]

d) Solve the differential equation in part **(b)** to find, to the nearest minute, the time it takes to fill the cylinder from empty to a height of 4 **metres.**

$\boxed{}$, $\boxed{-2\sqrt{H}-160\ln\left|80-\sqrt{H}\right|+C}$, $\boxed{t \approx 16}$

a) SETTING UP A MODEL

- IN FLOW $\frac{dV}{dt} = 2400$
- OUT FLOW $\frac{dV}{dt} = -kH^{\frac{1}{2}}$
- NET FLOW $\frac{dV}{dt} = 2400 - kH^{\frac{1}{2}}$



$V = 4800H$

$\frac{dV}{dH} = 4800$

RELATING VARIABLES H & V

$\Rightarrow \frac{dV}{dH} \times \frac{dH}{dt} = 2400 - kH^{\frac{1}{2}}$

$\Rightarrow 4800\frac{dH}{dt} = 2400 - kH^{\frac{1}{2}}$

$\Rightarrow \frac{dH}{dt} = \frac{1}{2} - \frac{k}{4800}H^{\frac{1}{2}}$

$\Rightarrow \frac{dH}{dt} = \frac{1}{2} - BH^{\frac{1}{2}}$ // $\left(B = \frac{k}{4800} = \text{CONSTANT}\right)$

AS REQUIRED

b) USING THE CONDITION GIVEN: $H=16$, $\frac{dH}{dt} = -120$

$\Rightarrow -120 = -k \times 16^{\frac{1}{2}}$ (OUTFLOW ONLY)

$\Rightarrow -120 = -4k$

$\Rightarrow k = 30$

$\Rightarrow B = \frac{k}{4800} = \frac{30}{4800} = \frac{1}{160}$

$\therefore \frac{dH}{dt} = \frac{1}{2} - \frac{1}{160}H^{\frac{1}{2}}$

$\frac{dH}{dt} = \frac{80 - H^{\frac{1}{2}}}{160}$ // AS REQUIRED

c) USING THE SUBSTITUTION GIVEN

$u = 80 - \sqrt{H}$

$\sqrt{H} = 80 - u$

$H = (80-u)^2$

$\frac{dH}{du} = 2(80-u)^1(-1)$

$\frac{dH}{du} = -2(80-u)$

$\frac{dH}{du} = 2(u-80)$

$dH = 2(u-80)\,du$

$\int \frac{1}{80-\sqrt{H}}\,dH = \int \frac{1}{u} \times 2(u-80)\,du$

$= \int \frac{2u-160}{u}\,du = \int 2 - \frac{160}{u}\,du$

$= 2u - 160\ln|u| + C$

$= 2(80-\sqrt{H}) - 160\ln|80-\sqrt{H}| + C$

$= 160 - 2\sqrt{H} - 160\ln|80-\sqrt{H}| + C$

$= -2\sqrt{H} - 160\ln|80-\sqrt{H}| + C$ //

d) SEPARATING VARIABLES

$\Rightarrow \frac{dH}{dt} = \frac{80-\sqrt{H}}{160}$

$\Rightarrow \int \frac{1}{80-\sqrt{H}}\,dH = \int \frac{1}{160}\,dt$

PART (c)

$\Rightarrow -2\sqrt{H} - 160\ln|80-\sqrt{H}| = \frac{1}{160}t + C$

APPLY CONDITION $t=0$ $H=0$ $\Rightarrow C = -160\ln 80$

$\Rightarrow -2\sqrt{H} - 160\ln|80-\sqrt{H}| = \frac{1}{160}t - 160\ln 80$

FINALLY WHEN $H = 4\text{m} = 400\text{cm}$

$\Rightarrow -2\times 20 - 160\ln(80-20) = \frac{1}{160}t - 160\ln 80$

$\Rightarrow \frac{1}{160}t = 160\ln 80 - 160\ln 60 - 40$

$\Rightarrow t \approx 964.66105\ldots$ seconds ≈ 16 minutes //